

### [6.33] Sine and cosine integrals

The sine and cosine integrals are functions which sometimes come up in integration and engineering applications. For example, the sine integral shows up when evaluating the radiated power of antennas, and some hypergeometric functions. The sine integral is called  $Si(z)$ , and the cosine integral is called  $Ci(z)$ . They are usually defined as

$$Si(z) = \int_0^z \frac{\sin(t)}{t} dt$$

$$Ci(z) = \gamma + \ln(z) + \int_0^z \frac{\cos(t)-1}{t} dt$$

or sometimes defined as

$$Si(z) = \frac{\pi}{2} - \int_z^\infty \frac{\sin(t)}{t} dt$$

$$Ci(z) = - \int_z^\infty \frac{\cos(t)}{t} dt$$

These symmetry relations hold:

$$Si(-z) = -Si(z)$$

$$Ci(-z) = Ci(z) - i\pi \quad \text{where } 0 < \arg(z) < \pi$$

where

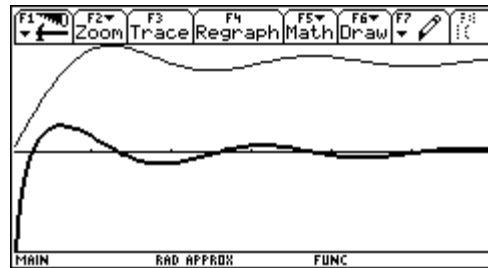
$$\gamma = 0.577215664901533 \text{ (Euler's constant)}$$

$Si(x)$  approaches  $\pi/2$  as  $x \rightarrow \infty$ , and  $Ci(x)$  approaches 0 as  $x \rightarrow \infty$ . These series expansions also calculate the integrals:

$$Si(z) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!} \right]$$

$$Ci(z) = \gamma + \ln(z) + \sum_{n=1}^{\infty} \left[ \frac{(-1)^n z^{2n}}{2n(2n)!} \right]$$

This plot shows the two functions from  $x = 0.1$  to  $x = 15$ . The thin plot trace is  $Si(x)$ , and the thick trace is  $Ci(x)$ .



These functions can be calculated with the routines  $Si()$  and  $Ci()$ , shown below. These routines are not terribly accurate, but are good enough for most engineering applications. For arguments less than  $\pi$ , the built-in  $nint()$  function is used to integrate the function definition. For arguments greater than  $\pi$ , a rational approximation is used. The transition point of  $\pi$  was selected more for reasons of execution time than anything else. Above that point,  $nint()$  takes significantly longer to execute.

The routines must be installed in a folder called `/spfn`. Set the modes to Approx, and Rectangular or Polar complex before execution. Input arguments must be real numbers, not complex numbers. Note that  $Ci(x)$  returns a complex number for arguments less than zero.

Because different methods are used for different argument ranges, there is a discontinuity in the function at  $x = \pi$ . The discontinuity is about  $2.6E-7$  for  $Si(x)$ , and about  $1.5E-8$  for  $Ci(x)$ . This may not

practically affect the results when simply evaluating the function, but it may be a problem if you numerically differentiate the functions, or try to find a maximum or minimum with the functions.

The rational approximations used for Si(x) and Ci(x) are based on the identities

$$\text{Si}(x) = \frac{\pi}{2} - f(x) \cos(x) - g(x) \sin(x)$$

$$\text{Ci}(x) = f(x) \sin(x) - g(x) \cos(x)$$

where

$$f(x) = \frac{1}{x} \left( \frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \varepsilon(x) \quad |\varepsilon(x)| < 5 \cdot 10^{-7}$$

$$\begin{array}{ll} a_1 = 38.027264 & b_1 = 40.021433 \\ a_2 = 265.187033 & b_2 = 322.624911 \\ a_3 = 335.677320 & b_3 = 570.236280 \\ a_4 = 38.102495 & b_4 = 157.105423 \end{array}$$

$$g(x) = \frac{1}{x^2} \left( \frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \varepsilon(x) \quad |\varepsilon(x)| < 3 \cdot 10^{-7}$$

$$\begin{array}{ll} a_1 = 42.242855 & b_1 = 48.196927 \\ a_2 = 302.757865 & b_2 = 482.485984 \\ a_3 = 352.018498 & b_3 = 1114.978885 \\ a_4 = 21.821899 & b_4 = 449.690326 \end{array}$$

All of these equations are from *Handbook of Mathematical Functions*, Abramowitz and Stegun, Dover, 1965.

The results from both of these routines are only accurate to about seven significant digits. Below  $x = \pi$ , the accuracy may be as good as 11 or 12 significant digits. Notice that the maximum argument for both routines is 1E12 radians. This limit results from the use of the built-in *sin()* and *cos()* functions.

The evaluation of *ci(x)* and *si(x)* is also complicated by the fact that *nInt()* evaluates the integrals very slowly, and with poor accuracy, for arguments less than about 1E-4. To get around this, I used Taylor series expansions for the integrands near  $x=0$ , then symbolically integrated those. For the cosine integral, I use

$$\frac{\cos(t)-1}{t} = -\frac{t}{2} + \frac{t^3}{24} - \frac{t^5}{720} + \frac{t^7}{40320} + \dots$$

and integrated this expansion to find

$$\text{Ci}(x) \approx -\frac{x^2}{4} + \frac{x^4}{96} - \frac{x^6}{4320} + \frac{x^8}{322560}$$

Similarly, for the sine integral:

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{6} + \frac{t^4}{120} - \frac{t^6}{5040} + \frac{t^8}{362880} - \frac{t^{10}}{39916800} + \dots$$

and integrating gives

$$\text{Si}(x) \approx x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \frac{x^9}{3265920} - \frac{x^{11}}{439084800}$$

These approximations give good accuracy for  $x < 0.01$ .

Code listing for si(x):

```
si(x)
Func
©(x) return si(x), x real, |x|<1E12
©Must be in folder \spfn
©24jan01/dburkett@infinet.com

if x=0:return 0
if x<0:return -spfn\si(-x)

if x>π then

π/2-(((polyeval({1,0,38.027264,0,265.187033,0,335.677320,0,38.102495},x))/(polyeval({1,0,40.021433,0,322.624911,0,570.23628,0,157.105423},x)))/(x)*cos(x)+(polyeval({1,0,42.242855,0,302.757865,0,352.018498,0,21.821899},x)/polyeval({1,0,48.196927,0,482.485984,0,1114.978885,0,449.690326},x))/(x^2)*sin(x))

elseif x>.01 then

nint(sin(z)/z,z,0,x)

else

polyeval({-2.2774643986765E-9,0,3.0619243582207E-7,0,-2.8344671201814E-5,0,1.666666666667E-3,0,-.055555555555556,0,1,0},x)

endif

EndFunc
```

Code listing for ci(x):

```
ci(x)
Func
©(x) return ci(x), x real, |x|<1E12
©Must be in folder \spfn
©23jan01/dburkett@infinet.com

if x=0:return undef
if x<0:return spfn\ci(-x)+i*π

if x>π then

(((polyeval({1,0,38.027264,0,265.187033,0,335.677320,0,38.102495},x))/(polyeval({1,0,40.021433,0,322.624911,0,570.23628,0,157.105423},x)))/(x))*sin(x)-((polyeval({1,0,42.242855,0,302.757865,0,352.018498,0,21.821899},x)/polyeval({1,0,48.196927,0,482.485984,0,1114.978885,0,449.690326},x))/(x^2))*cos(x))

elseif x>.01 then

nint((cos(t)-1)/t,t,0,x)+ln(x)+.57721566490153

else

ln(x)+.57721566490153+polyeval({3.100198412698E-6,0,-2.3148148148148E-4,0,.010416666666667,0,-.25,0,0},x)

endif

EndFunc
```

*(Thanks to Bhuvanesh for providing test data.)*